MATH 140A Review: Helpful Algebraic techniques

Facts to Know:

Remember algebraic tricks from the past:

(Multiply by the conjugate) Remember the *difference of two squares*:

$$x^2 - y^2 = (x+y)(x-y)$$

("$e^{\ln}$ trick") Remember properties of logarithms:

$$x = e^{\ln x}$$

Examples: Use algebraic techniques to compute the limit of a_n as $n \to \infty$:

1. $a_n = \left(\sqrt{n+1} - \sqrt{n-1}\right) \cdot \frac{\left(\sqrt{n+1} + \sqrt{n-1}\right)}{\sqrt{n+1} + \sqrt{n-1}} = \frac{(n+1) - (n-1)}{\sqrt{n+1} + \sqrt{n-1}}$

$$= \frac{2}{\sqrt{n+1} + \sqrt{n-1}} \longrightarrow 0,$$

as $n \to \infty$.

Thus, $a_n \to 0$ as $n \to \infty$.

2. $a_n = \frac{n+1}{n^2+2} \cdot \frac{1/n^2}{1/n^2} = \frac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{2}{n^2}} \xrightarrow{n\to\infty} \frac{0}{1}.$

Thus, $a_n \to 0$ as $n \to \infty$.

3. $a_n = \frac{1/n \cdot \ln n}{1} = \frac{\ln n}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

4. $a_n = n^{1/n} = e^{\ln(n^{1/n})} = e^{\frac{1}{n}\ln n}.$

Since $\frac{1}{n}\ln n \longrightarrow 0$ as $n\to\infty$ and e^x is continuous function, we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{\lim_{n\to\infty}\frac{1}{n}\ln n} = e^0 = 1.$$